APPLIED ANALYSIS/NUMERICAL ANALYSIS
QUALIFYING EXAMINATION
August 2009

Part 1: Applied Analysis

Work 3 out of 4 problems of this part of the exam.

Policy on Misprints. *The qualifying examination committee tries to proofread the examinations as carefully as possible. Nevertheless, there may be a few misprints. If you are convinced that a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.*

Q1. Let L be the Sturm-Liouville operator

$$L = \frac{d}{dx}[p(x)\frac{d}{dx}] + q(x)$$

and let D be the boundary conditions operator

$$Du = \begin{cases} \alpha_1 u'(0) + \beta_1 u(0) \\ \alpha_2 u'(l) + \beta_2 u(l) \end{cases}$$

Let $f(x)$ be a continuous function on $[0, l]$ and assume that the problem $Lu = f, \;\; Du = 0$ is regular, i.e., the homogeneous problem has only the trivial solution.

(a) List the properties of a Green's function $g(x, y)$ which satisfies the 2^{th} order equation

$$L_x g(x, y) = \delta(x - y)$$

in the sense of distributions.

(b) Given the existence of a Green's function, write down a solution to

$$L_x u(x) = f(x), \;\; x \in [0, l].$$

(c) Describe for which complex λ one can find a Green's function for the differential operator

$$Lu = u'' + \lambda u, \;\; u \in L^2(-\infty, \infty)$$

and for those λ, find the Green's function.

Q2. (a) State the Fredholm Alternative Theorem for bounded linear operators on Hilbert space.

(b) Consider the integral equation

$$(Lu)(t) = u(t) + \lambda \int_0^1 su(s)ds = f(t).$$

where $u, f \in L^2[0,1]$.
Explain why L has closed range for all λ, find the values of λ for which L is invertible and write an expression for L^{-1}. (Hint: given f, "guess" the solution u.)

(c) For those λ where L is not invertible, describe under what conditions one can solve $Lu = f$ and how one might do this.

Q3. (a) State the Contraction Mapping Theorem.

(b) Prove that the Fredholm integral equation $x = Fx$ has a solution where

$$(Fx)(t) = \int_0^1 K(s,t,x(s))ds + w(t), \;\; t \in [0,1]$$

and where

i. x and w are in $C[0,1]$, $\|x\| = max_t|x(t)|$.

ii. $K(s,t,r)$ is continuous on $0 \le s,t \le 1, -\infty < r < \infty$

iii. $|K(s,t,\xi) - K(s,t,\eta)| \le \theta|\xi - \eta|, \;\; 0 < \theta < 1$.

(c) Is the solution to (b) unique? Prove or disprove.

(d) Describe an iteration procedure to numerically solve the equation in (b).

Q4. Let $S^h(3,1)$ denote the finite element space of cubic splines on $[0,1]$. The space $S^h(3,1)$ is spanned by two sets of cubic polynomials

$$\phi_j(x) = \phi(\frac{x - x_j}{h}), \quad \psi_j(x) = h\psi(\frac{x - x_j}{h}),$$

for $j = 0, 1, 2, \cdots, N$ where $h = \frac{1}{N}, \;\; x_j = \frac{j}{N}$ and

$$\phi(x) = (|x| - 1)^2(2|x| + 1), \quad \psi(x) = x(|x| - 1)^2.$$

(a) Define linear projection on the space $C[0,1]$.

(b) Let $\phi_k(x), k = 0, \cdots, N$ be the piecewise linear finite element basis functions satisfying $\phi_k(j/N) = \delta_{j,k}$. Show that

$$Pf = \sum_{j=0}^{N} f(j/N)\phi_j(x)$$

is a projection on the space of continuous functions $C[0,1]$.

(c) Define a projection on $C^1[0,1]$, the space of continuously differentiable functions, using cubic splines.

APPLIED/NUMERICAL ANALYSIS QUALIFIER: NUMERICAL ANALYSIS PART

August 13, 2009

Problem 1 Consider the following finite element triple:

- K =a rectangle with vertices $\{a^i\}$, $i = 1, 2, 3, 4$.
- $P = Q^3 = span\{x_1^i x_2^j \ ; \ i, j = 0, \ldots, 3\}$.
- $N = \{p(a^i), p_1(a^i), p_2(a^i), p_{12}(a^i), \ i = 1, 2, 3, 4\}$. (Here p_i denotes differentiation with respect to x_i).

(a) Show that the above finite element is unisolvent.

(b) What do you need to do to check if the above element with a rectangular mesh results in a C^1 finite element space?

(c) Does the above element (with a rectangular mesh) result in a C^1 finite element space? (Explain your answer).

Problem 2 Consider the Neumann Problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} &= g \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.1}$$

Here Ω is a bounded domain in $\mathbb{R}^2$ and f and g are suitably smooth.

(a) Derive a weak form of the above problem using a test function in $H^1(\Omega)$.

(b) Discuss when the weak form of Part (a) has a solution and if it is unique.

(c) Describe a variational formulation of (2.1) in terms of an appropriate Hilbert space V. Be sure to explicitly define V.

(d) Prove coercivity of the form of Part (a) on the V of Part (c) when $\Omega = (0, 1)^2$.

Problem 3 Let $\Omega_e = \{x \in \mathbb{R}^2 : \ \|x\| > 1\}$. Show that the Poincaré inequality does not hold in $H_0^1(\Omega_e)$, i.e., there does not exist a constant $c > 0$ satisfying

$$c\|u\|^2_{L^2(\Omega_e)} \leq \int_{\Omega_e} \|\nabla u\|^2 \, dx \quad \text{for all} \quad u \in H_0^1(\Omega_e).$$

The space $H_0^1(\Omega_e)$ is the completion of $C_0^\infty(\Omega^c)$ in the norm

$$\|v\|_{H^1(\Omega^c)} = \left(\|v\|^2_{L^2(\Omega^c)} + \|\nabla v\|^2_{(L^2(\Omega^c))^2} \right)^{1/2}.$$

(Hint: Consider dilating a fixed function.)