# Applied/Numerical Analysis Qualifying Exam

August 13, 2021

## Cover Sheet – Applied Analysis Part

**Policy on misprints:** The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____________________________________________________

**Combined Applied Analysis/Numerical Analysis Qualifier**
**Applied Analysis Part**
**August 13, 2021**

**Instructions:** *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

**Problem 1.** Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Give an example of a function in $\mathcal{D}$.
(c) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = x^2\phi(x)$ for some $\phi \in \mathcal{D}$ if and only if $\psi(0) = 0$ and $\psi'(0) = 0$.
(d) Use 2(c) to find all $T \in \mathcal{D}'$ that satisfy $x^2T(x) = 0$.

**Problem 2.** Let $\mathcal{P}$ be the set of all polynomials.

(a) State and sketch a proof of the Weierstrass approximation theorem.
(b) let $\mathcal{H} = L^2_w[0,1]$, where the inner product is $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}w(x)dx$ and where $w \in C[0,1]$, $w(x) \geq c > 0$ on $[0,1]$. Show that $\mathcal{P}$ is dense in $L^2_w[0,1]$. (You may use the density of $C[0,1]$ in $L^2[0,1]$.)
(c) Let $\mathcal{U} := \{p_n\}_{n=0}^{\infty}$ be the orthonormal set of polynomials obtained from $\mathcal{P}$ via the Gram-Schmidt process. Show that $\mathcal{U}$ is a complete set in $L^2_w[0,1]$.

**Problem 3.** Suppose that $K$ is a compact operator and $Tu(x) := \int_{-\infty}^{\infty} e^{-|x-y|^2}u(y)dy$.

(a) Show that if $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal set, then $\lim_{j\to\infty} K\phi_j = 0$.
(b) Show that $T$ is a bounded operator on $L^2(\mathbb{R})$.
(c) The set $\phi_j = \chi_{[j,j+1]}$ is an orthonormal basis for $L^2(\mathbb{R})$. Use translation invariance to show that $\|T\phi_j\| = \|T\phi_0\|$.
(d) Show that $T$ is *not* compact.

**Problem 4.** Let $Lu = x^2u'' + 2xu' - 2u$. $D_L = \{u \in L^2[1,2]],\ u'(1) = 0, u(2) = 0\}$. You are given that $x, x^{-2}$ are homogenous solutions.

(a) Show that $L = L^*$.
(b) Find the Green's function $G$. Show that $Ku(x) = \int_1^2 G(x,y)u(y)dy$ is compact and selfadjoint.
(c) State the spectral theorem for compact, self-adjoint operators. Use it to show that the (normalized) eigenfunctions of the eigenvalue problem $Lu + \lambda u = 0$, $u \in D_L$, form a complete orthonormal set in $L^2[1,2]$.

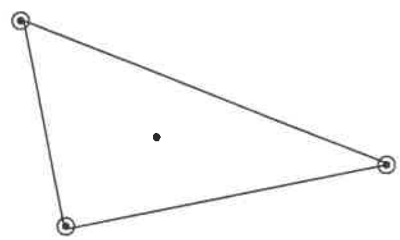
## NUMERICAL ANALYSIS QUALIFIER

August, 2021

**Problem 1.** Consider the $C^0$ Hermite cubic finite element $(T, \mathbb{P}^3, \Sigma)$, where

$T \subset \mathbb{R}^2$ is a triangle with vertices $v_1, v_2, v_3$ and barycenter $b$,
$\mathbb{P}^3$ is the set of polynomials of degree 3 or less on $T$,
$\Sigma = \{p(v_i), p(b), \nabla p(v_i),\ i = 1, 2, 3\}$.

The diagram for the degrees of freedom is below (dots are function evaluations, circles are gradient evaluations). Show that $(T, \mathbb{P}^3, \Sigma)$ is a finite element.



*Hints:*

(1) You may assume without loss of generality that $T$ is the unit triangle if you wish.
(2) Each gradient evaluation $\nabla p(v_i)$ yields *two* degrees of freedom which may be taken to be any two directional derivatives in independent directions.
(3) You may use the following elementary factorization result without proof:

*If $p \in \mathbb{P}^k$ and $p = 0$ on the line $L(x, y) = 0$, then $p = Lp_{k-1}$ with $p_{k-1} \in \mathbb{P}^{k-1}$.*

**Problem 2.** Consider the boundary value problem

$$-u''(x) - \alpha u(x) = f(x), \quad 0 < x < 1, \qquad u(0) = 0,\ u'(1) = 0, \tag{2.1}$$

where $f(x)$ is a given function on $(0, 1)$ and $\alpha > 0$ is a given constant. *Note:* Below you may assume:

(1) The validity of an appropriate Poincaré inequality.
(2) Approximation (interpolation) error bounds for the finite element spaces you define below.

However, be sure to correctly and clearly state these results with appropriate hypotheses before using them.

(a) Give a weak formulation of this problem. As part of deriving the weak formulation, be sure to define an appropriate variational space $V$.

(b) Prove that the corresponding bilinear form is coercive on $V$. This result is only valid for a restricted range of values of the parameter $\alpha$. Clearly state for which $\alpha$ your result holds, and explicitly include dependence on $\alpha$ and the Poincaré constant in your coercivity constant.

(c) Set up a finite dimensional space $V_h \subset V$ of piece-wise polynomial functions of degree $k$ over a uniform partition of $(0, 1)$. Introduce the Galerkin finite element method for the problem (2.1) for $V_h$. State (but do not prove) an error estimate in the $V$-norm assuming that $u \in H^{k+1}(0, 1)$.

(d) Assuming "full regularity" and using a duality argument **prove** the following estimate for the error of the Galerkin solution $u_h$:

$$\|u - u_h\|_{L^2} \le C h^{k+1} \|u^{(k+1)}\|_{L^2}. \tag{2.2}$$

**Problem 3.** Let $\Omega$ be a bounded domain and $T > 0$ be a given final time. For $f \in C^0([0,T]; L_2(\Omega))$ and $u_0 \in H_0^1(\Omega)$ given, we consider the parabolic problem consisting in finding $u : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) - \Delta u(x,t) = f(x,t) & \text{for } (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

We assume that the solution $u$ to the above problem is sufficiently smooth.

Let $N$ be a strictly positive integer and let $\tau := T/N$, $t_n := n\tau$ and $t^{n+\frac{1}{2}} := \frac{1}{2}(t^{n+1} + t^n)$ for $n = 0, \dots, N$. We consider the following semi-discretization in time: Set $U^0 := u_0$ and define $U^n \in H_0^1(\Omega)$ recursively by

$$\begin{cases} \frac{1}{\tau}(U^{n+1}(x) - U^n(x)) - \frac{1}{2}\Delta(U^{n+1}(x) + U^n(x)) = f(x, t^{n+\frac{1}{2}}) & \text{for } x \in \Omega, \\ U^{n+1}(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

(a) (Stability) Show that for $n = 0, \dots, N$, $U^n$ satisfies

$$\|U^{n+1}\|_{L_2(\Omega)} \le \|U^0\|_{L_2(\Omega)} + \tau \sum_{j=0}^{n} \|f(t^{j+\frac{1}{2}})\|_{L_2(\Omega)}. \tag{3.1}$$

<u>Hint:</u> Write the time-discretized problem in weak form before multiplying by a suitable test function.

(b) (Consistency I) Show either (but not both) that

$$\|\frac{1}{\tau}(u(t^{n+1}) - u(t^n)) - \frac{\partial}{\partial t} u(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \le C\tau \|\frac{\partial^3}{\partial t^3} u\|_{L_1(t^n, t^{n+1}; L_2(\Omega))}$$

or

$$\|\frac{1}{2}\Delta\left(u(t^{n+1}) + u(t^n)\right) - \Delta u(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \le C\tau \|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_1(t^n, t^{n+1}; L_2(\Omega))}.$$

Here $C$ is a constant independent of $\tau$, $T$ and $u$.

<u>Hint:</u> You can use without proof the following Taylor expansion formula

$$g(b) = g(a) + g'(a)(b-a) + \dots + \frac{1}{n!} g^{(n)}(a)(b-a)^n + \frac{1}{n!} \int_a^b (b-t)^n g^{(n+1)}(t)\,dt.$$

(c) (Consistency II) Deduce from the previous item that for a constant $C$ independent of $\tau$, $T$ and $u$ we have

$$\begin{aligned} &\|\frac{1}{\tau}(u^{n+1}(x) - u^n(x)) - \frac{1}{2}\Delta(u^{n+1}(x) + u^n(x)) - f(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \\ &\le C\tau \left( \|\frac{\partial^3}{\partial t^3} u\|_{L_1(t^n, t^{n+1}; L_2(\Omega))} + \|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_1(t^n, t^{n+1}; L_2(\Omega))} \right). \end{aligned} \tag{3.2}$$

(d) From (3.1) and (3.2), conclude the following estimate for the error $e^n := u(t^n) - U^n$:

$$\|e^N\|_{L_2(\Omega)} \le C\tau^2 \left( \|\frac{\partial^3}{\partial t^3} u\|_{L_1(0,T; L_2(\Omega))} + \|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_1(0,T; L_2(\Omega))} \right),$$

where $C$ is a constant independent of $\tau$, $T$ and $u$.