Applied/Numerical Analysis Qualifying Exam

January 12, 2016

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
January 12, 2016

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Recall that the DFT and inverse DFT are given by $\hat{y}_k = \sum_{j=0}^{n-1} y_j \bar{w}^{jk}$ and $y_j = \frac{1}{n}\sum_{j=0}^{n-1} \hat{y}_k w^{jk}$, where $w = e^{2\pi i/n}$.

(a) State and prove the Convolution Theorem for the DFT.

(b) Let a, x, y be column vectors with entries $a_0, \dots, a_{n-1}$, $x_0, \dots, x_{n-1}$, $y_0, \dots, y_{n-1}$. In addition, let α, ξ and η be n-periodic sequences, the entries for one period, $k = 0, \dots, n-1$, being those of a, x, and y, respectively. Consider the circulant matrix

$$A = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}.$$

Show that the matrix equation $Ax = y$ is equivalent to convolution $\eta = \alpha * \xi$.

(c) Use parts (a) and (b) above to find the eigenvalues of

$$A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

Problem 2. Let $Lu = -(e^x u')'$, $u(0) = 0$, $u'(1) = 0$.

(a) Find the Green's function $G(x, y)$ for $Lu = -(e^x u')' = f$, $u(0) = 0$, $u'(1) = 0$.

(b) Why is $Kf(x) = \int_0^1 G(x, y) f(y) dy$ compact? (One sentence will do.)

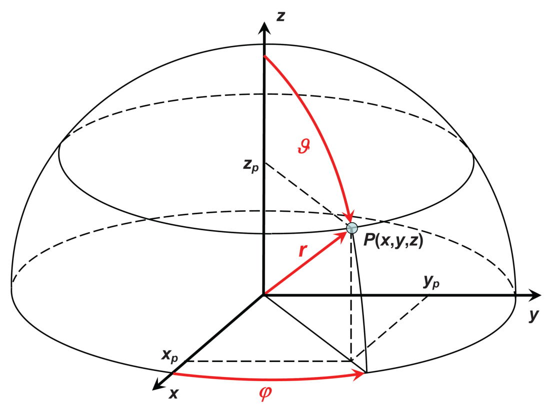
(c) Consider the eigenvalue problem $Lu = \lambda u$, $u(0) = 0$, $u'(1) = 0$. Show that the (orthonormal) set of eigenfunctions for L form a complete set in $L^2[0, 1]$.

Problem 3. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) State and prove the Closed Range Theorem.

(b) Let $\mathcal{H} = L^2[0, 1]$. Define the kernel $k(x, y) := x^2 y^9$ and let $Ku(x) = \int_0^1 k(x, y) u(y) dy$. Show the K is in $\mathcal{C}(L^2[0, 1])$.

(c) Let $L = I - \lambda K$, $\lambda \in \mathbb{C}$, with K as defined in part (b) above. Find all λ for which $Lu = f$ *can* be solved for all $f \in L^2[0, 1]$. For these values of λ, find the resolvent $(I - \lambda K)^{-1}$.



Problem 4. Recall that a *geodesic* on a surface provides the path of shortest distance between two points on a surface. Let S be the unit sphere in $\mathbb{R}^3$. In the coordinates shown above, the differential arc length is given by $ds = \sqrt{d\theta^2 + \sin^2(\theta) d\varphi^2}$. If $P_0 = (\theta_0, 0)$ and $P_1 = (\theta_1, 0)$, $0 < \theta_0 < \theta_1 < \pi$, show that the geodesic is the arc of the great circle given by $\theta_0 \leq \theta \leq \theta_1$, $\varphi = 0$. Hint: describe curves joining the two points by $\varphi = u(\theta)$, where $u \in C^2[\theta_0, \theta_1]$ and satisfies $u(\theta_0) = u(\theta_1) = 0$. Minimize the arc-length functional.

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Name____

NUMERICAL ANALYSIS PART

January 12, 2016

Problem 1. *Let b be a strictly positive constant and consider the problem: find $u(x,t)$ such that*

$$\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0, \quad 0 < x < 1,\ 0 < t$$
$$u(x,0) = u_0(x), \quad 0 < x < 1,$$
$$u(0,t) = u(1,t),\ t > 0$$

where u_0 is a smooth function. Let J and N be positive integers, $x_i = ih$ where $h = 1/J$ and $t_n = n\tau$ where $\tau = 1/N$. Also denote by u_j^n the approximation of $u(x_j, t_n)$.

Set $u_j^0 = u_0(x_j)$ and define reccursively u_j^n by the following Lax-Friedrichs *scheme*

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\tau b}{2h}(u_{j+1}^n - u_{j-1}^n), \quad j = 1, ..., J.$$

Show that for all $j = 1, ..., J$ and $n \geq 0$

$$\min_i(u_i^0) \leq u_j^n \leq \max_i(u_i^0)$$

provided $\frac{\tau b}{h} \leq 1$.

Problem 2. *Below, C_i, for $i = 1, 2, 3$ denote positive constants. For $f \in L^2(\Omega)$, we consider solutions $u \in H^1(\Omega)$ to*

$$A(u,\phi) = \int f\phi, \quad \textit{for all} \ \ \phi \in H^1(\Omega). \tag{2.1}$$

Here Ω is a polyhedral domain in $\mathbb{R}^n$ and $A(\cdot,\cdot)$ is a (non-coercive) bounded bilinear form on $H^1(\Omega)$. It is assumed that A satisfies a Gärding inequality, i.e., there are positive constants K and α satisfying

$$\alpha\|v\|^2_{H^1(\Omega)} \leq A(v,v) + K\|v\|^2_{L^2(\Omega)}, \quad \textit{for all} \ \ v \in H^1(\Omega). \tag{2.2}$$

We assume that solutions of (2.1) and those of the adjoint problem: $u \in H^1(\Omega)$ satisfying

$$A(\phi,u) = \int_\Omega f\phi, \quad \textit{for all} \ \ \phi \in H^1(\Omega), \tag{2.3}$$

exist, are unique and satisfy

$$\|u\|_{H^2(\Omega)} \leq C_1\|f\|_{L^2(\Omega)}.$$

We finally assume that $\{V_h\}$, $h \in (0,1]$ is collection of conforming finite element subspaces satisfying the standard approximation properties and consider the finite element approximation: $u_h \in V_h$ satisfying

$$A(u_h,\theta) = \int_\Omega f\theta, \quad \textit{for all} \ \ \theta \in V_h. \tag{2.4}$$

(a) *Suppose that* u *solves* (2.1) *and* $u_h \in V_h$ *satisfies* (2.4) *(we do not assume that* u_h *is unique). Show that*

$$\|u - u_h\|_{L^2(\Omega)} \le C_2 h \|u - u_h\|_{H^1(\Omega)}.$$

(b) *Use* (2.2) *and Part (a) to show that there is an* $h_0 > 0$ *such that if* $h \le h_0$*,*

$$\frac{\alpha}{2}\|u - u_h\|^2_{H^1(\Omega)} \le A(u - u_h, u - u_h).$$

(c) *Use Part (b) to show that the solutions of* (2.4) *are unique when* $h \le h_0$*. This also implies existence.*

(d) *Prove that the unique solution (when* $h \le h_0$*) of* (2.4) *satisfies*

$$\|u - u_h\|_{H^1(\Omega)} \le C_3 \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

Problem 3. For this problem, for $M \ge 1$, S_M is a finite dimensional subspace of $H^2(\Omega)$ with $\Omega = (0,1)$. Also, we are given linear operators, $P_c : H^2(\Omega) \to S_M$ and $P_M : L^2(\Omega) \to S_M$. We further assume that there is a constant C_1 not depending on M, u or s and satisfying

$$|(I - P_M)u|_{H^s(\Omega)} \le C_1 M^{s-2} |u|_{H^2(\Omega)}, \quad \text{for all } u \in H^2(\Omega),\ s = \{0,1,2\}.$$

Here $|\cdot|_{H^s(\Omega)}$ denotes the $H^s(\Omega)$ semi-norm. We set $\Omega_M = (0, M)$. For u defined on Ω, we define $\hat{u}(x)$ for $x \in \Omega_M$ by $\hat{u}(x) = u(x/M)$ and define

$$\widehat{P}_M(\hat{u}) = \widehat{P_M u} \quad \text{and} \quad \widehat{P}_c(\hat{u}) = \widehat{P_c u}.$$

We finally assume there is a constant C_2 (not depending on M) satisfying

$$\|\widehat{P}_c \hat{u}\|_{L^2(\Omega_M)} \le C_2 \|\hat{u}\|_{H^2(\Omega_M)}, \quad \text{for all } \hat{u} \in H^2(\Omega_M),$$

and that $\widehat{P}_c \widehat{P}_M = \widehat{P}_M$.

(a) Derive a relationship between $|u|_{H^s(\Omega)}$ and $|\hat{u}|_{H^s(\Omega_M)}$.

(b) Show that there is a constant C_3 not depending on M satisfying

$$\|(I - \widehat{P}_M)\hat{u}\|_{H^2(\Omega_M)} \le C |\hat{u}|_{H^2(\Omega_M)}.$$

(c) Show that there is a constant C_3 not depending on M satisfying

$$\|(I - P_c)u\|_{L^2(\Omega)} \le C_3 M^{-2} |u|_{H^2(\Omega)}, \quad \text{for all } u \in H^2(\Omega).$$