Complex analysis qualifying exam, January 2018.

1. Give the statements of the following theorems:

   (a) Montel's theorem;
   (b) Runge's theorem.

2. Let f be holomorphic in the unit disk $\mathcal{D}$ and continuous in the closure of $\mathcal{D}$. Prove that

$$\int_0^1 f(x)dx = \frac{1}{2\pi i}\int_{\{|z|=1\}} f(z)\log z dz,$$

where log denotes the branch of logarithm whose imaginary part takes values in $(0, 2\pi)$.

3. Suppose that there exists a conformal map from $\{0 \le r_1 < |z| < r_2\}$ to $\{0 \le R_1 < |z| < R_2\}$. Prove that $r_1/r_2 = R_1/R_2$.

4. Does there exist a bounded holomorphic function in the unit disk $\mathcal{D}$ such that

$$f\left(1 - \frac{1}{n}\right) = \frac{(-1)^n}{n}$$

for $n = 1, 2, 3, \ldots$?

5. Calculate the integral

$$\int_{|z|=5} \frac{z}{e^z - i} dz.$$

6. Find the disk of convergence for the power series

$$f(z) = 1 + z^2 + z^{2^2} + \ldots + z^{2^n} + \ldots$$

Show that $f(z)$ cannot be analytically continued to any connected domain properly containing the disk of convergence.

7. Show that the range of the entire function $\frac{\sin z}{z}$ is the whole complex plane $\mathbb{C}$.

8. Find a conformal map from the disk with a slit $\{|z| < 1\} \setminus [0, 1]$ to the strip $\{|\text{Re } z| < 1\}$.

9. Let f be a holomorphic function in a complex domain such that f is not identically zero and for any positive integer n there exists a holomorphic function g in the same domain satisfying $g^n = f$. Prove that there exists a holomorphic function h in the same domain such that $e^h = f$.

10. Let $1 < a < \infty$. Prove that the function $e^z - z - a$ has exactly one zero in the left half-plane $\{\text{Re } z < 0\}$.