Notation. Let $\mathbb{R}^n$ denote real n-space. Let $S^n = \{x \in \mathbb{R}^{n+1} \ : \ |x| = 1\}$ denote the n-sphere in $\mathbb{R}^{n+1}$.

Employ the summation convention: any repeated index appearing as a subscript and superscript is summed over.

1.) Prove the *Tube Lemma*: Let X be a topological space and let Y be a compact topological space. Prove that if W is an open subset of $X \times Y$ (in the product topology) containing the "slice "$\{x_0\} \times Y$ for some $x_0 \in X$, then there exists an open neighborhood U of x_0 in X such that $U \times Y \subset W$.

2.) **(a)** Prove that an arcwise connected topological space is connected.

(b) Let X be a normal, connected topological space containing more than one point. Prove that X is uncountable. (*Hint*: Use Urysohn's lemma).

3.) Let $\mathbb{R}_\ell$ denote the real line $\mathbb{R}$ with the lower limit topology (so-called Sorgenfrey line), i.e. the topology consisting of all left-closed, right-open intervals $[a, b)$. Let $\mathbb{R}^2_\ell$ denote the plane $\mathbb{R}^2$ with the product topology $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

(a) Find the closure of the set (a, b) in $\mathbb{R}_\ell$.

(b) Prove that $\mathbb{R}_\ell$ is not a locally compact space.

(c) Take the 'anti-diagonal' $L = \{(x, -x) \ : \ x \in \mathbb{R}_\ell\}$ of $\mathbb{R}^2_\ell$. Describe the subspace topology that L inherits from $\mathbb{R}^2_\ell$.

(d) Prove that the space $\mathbb{R}^2_\ell$ is not Lindelöf (recall that a topological space X is called Lindelöf if every open cover of X has a countable subcover).

4.) Let $\mathbb{R}/\mathbb{Z}$ be the quotient space obtained from $\mathbb{R}$ by the identification of the subspace $\mathbb{Z}$ to a point. (Do not confuse this with the group quotient of $\mathbb{R}$ by $\mathbb{Z}$.) Prove that this quotient space is not first countable.

5.) **(a)** Prove that for any locally finite family of sets $\{V_\alpha\}_{\alpha \in A}$ in a topological space X one has

$$\overline{\bigcup_{\alpha \in A} V_\alpha} = \bigcup_{\alpha \in A} \overline{V_\alpha}\,.$$

(b) Recall that a topological space X is called countably compact if every countable cover of X has a finite subcover. Prove that a countably compact paracompact space is compact.

6.) **(a)** State the definition of a smooth n–dimensional manifold.

(b) Set $A = \mathbb{R}^{n+1} \backslash \{0\}$. Given $x, y \in A$, the condition

$$x \sim y \textit{ if and only if there exists } \lambda \neq 0 \textit{ such that } y = \lambda x$$

defines an equivalence relation on A. *Real projective space* $\mathbb{RP}^n$ is the set of equivalence classes $A/\sim$. (Equivalently, $\mathbb{RP}^n$ is the set of lines in $\mathbb{R}^{n+1}$ through the origin.) Prove that $\mathbb{RP}^n$ admits the structure of a smooth n–dimensional manifold.

7.) **(a)** State the Implicit Function Theorem.

(b) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$(x, y, z) \mapsto (xz - y^2 \, , \, yz - x^2) \, .$$

For which values $(a, b) \in \mathbb{R}^2$ is $F^{-1}(a, b)$ a smooth submanifold of $\mathbb{R}^3$?

8.) Let (s, t) be coordinates on $\mathbb{R}^2$ and (w, x, y, z) be coordinates on $\mathbb{R}^4$. Define $F : \mathbb{R}^2 \to \mathbb{R}^4$ by

$$F(s, t) = (s^2 \, , \, 7\,st \, , \, t^3 \, , \, s - t - 1) \, .$$

(a) Compute the push-forward $F_*(\frac{\partial}{\partial s} + \frac{\partial}{\partial t})$.

(b) Compute the pull-back $F^*(\mathrm{d}x + y\,\mathrm{d}z)$.

9.) Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < z = x^2 + y^2\}$. Compute the Gauss and mean curvatures of S.

10.) Set $N = (0, 1) \in S^1 \subset \mathbb{R}^2$, and $U = S^1 \backslash \{N\}$. Stereographic projection defines a coordinate chart $\varphi : U \to \mathbb{R}$ by

$$\varphi(x, y) = \frac{x}{1 - y} \, .$$

Let t be the Cartesian coordinate on $\mathbb{R}^1$. Characterize those functions $f : \mathbb{R} \to \mathbb{R}$ for which the vector field $F = \varphi_*^{-1}(f(t)\frac{\partial}{\partial t})$, defined on U, extends to a smooth vector field on S^1.