GEOMETRY AND TOPOLOGY QUALIFIER

Instructions. Answer all questions. Write your name and page number in the upper right corner of each page. Start each problem on a new sheet of paper, and use only one side of each sheet.

1. Let X be a compact metric space. Prove that (a) every real valued continuous function on X has a maximum value; (b) every continuous function on X is uniformly continuous.

2. Let S^1 be the circle $S^1 = \{e^{i2\pi t} : t \in \mathbb{R}\}$. Define an equivalence relation $\sim$ on S^1 by: $e^{i2\pi t_1} \sim e^{i2\pi t_2}$ iff $t_1 - t_2$ is an integer multiple of $\sqrt{2}$. Prove that the quotient space $X = S^1/\sim$ is not Hausdorff and find all continuous functions on X.

3. Let $SL_n(\mathbb{R})$ be the $n \times n$ matrices with real entries and determinant 1. Prove that $SL_n(\mathbb{R})$ is a noncompact smooth manifold and find the dimension of the manifold.

4. Let $X_1 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ and $X_2 = -x_2^2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$ be two smooth vector fields on $\mathbb{R}^2$. Is there a coordinator chart (y, U) for a neighborhood U of $(1,1)$ such that $X_1 = \frac{\partial}{\partial y_1}$ and $X_2 = \frac{\partial}{\partial y_2}$? Prove your claim.

5. Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be given the standard Riemannian metric inherited from $\mathbb{R}^3$. The projective space $\mathbb{P}^2$ is obtained from S^2 by identifying all antipodal points, i.e. $\mathbb{P}^2 = S^2/\sim$, where two points $p \sim q$ iff $p = -q$. Let π be the canonical map from S^2 to $\mathbb{P}^2$ by mapping each point to its equivalence class.

(a) prove that $\mathbb{P}^2$ is a compact smooth manifold;

(b) prove that there exists a unique Riemannian metric on $\mathbb{P}^2$ such that its pullback (by π) to S^2 is the standard Riemannian metric on S^2;

(c) find all geodesics on $\mathbb{P}^2$ with respect to the Riemannian metric in (b) and prove that every two distinct geodesics intersect exactly once;

(d) compute the curvature of the above Riemannian metric on $\mathbb{P}^2$.

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