Geometry and Topology Qualifier

Name: ____________________

This exam has 8 questions, for a total of 120 points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
8	15	
Total:	120	

Problem 1. (15 pts)

(a) Let X be a Hausdorff topological space. If $\{x_n\}$ is a convergent sequence in X, prove that $lim_{n \to \infty} x_n$ is unique.

(b) If the Hausdorff condition in part 1 is dropped. Is the same conclusion still true? Prove your claim if your answer is yes. Give a counter example if your answer is no.

Problem 2. (15 pts)

Let the real line R be given the standard topology. Construct a topology on R^2 such that every function from R^2 to R is continuous. Justify your answer.

Problem 3. (15 pts)

Let X be the connected sum of the torus with the Klein bottle. Compute the fundamental group of X.

Problem 4. (15 pts)

Let D^2 be the closed disk in the plane. Prove that any continuous map $f : D^2 \to D^2$, has a fixed point, i.e. there exists $p \in D^2$ satisfying $f(p) = p$.

Problem 5. (15 pts)

(a) Suppose M is a smooth n-dimensional manifold and $p \in M$. Define the tangent space T_pM of M at p.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $f(x, y) = (x, xy)$. Compute the pushforward $f_*(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})$.

(c) Determine the regular values of the map f in the previous part.

Problem 6. (15 pts)

Let $O(n)$ denote the set of $n \times n$ matrices A so that $AA^\intercal = I$, where I is the $n \times n$ identity matrix.

(a) Show that $O(n)$ is a Lie group.

(b) Find the Lie algebra of $O(n)$ and determine its dimension.

Problem 7. (15 pts)

(a) Let Δ be the distribution $\Delta = \ker(dz - y\,dx)$ on $\mathbb{R}^3$, i.e., $\vec{v} \in \Delta_{(x,y,z)}$ if $(dz - y\,dx)(\vec{v}) = 0$. Determine whether Δ is an involutive distribution.

(b) Let α be a 1-form on $\mathbb{R}^3$ and suppose that $\Delta = \ker(\alpha)$ is a distribution on $\mathbb{R}^3$. Suppose further that $\alpha \wedge d\alpha = dx \wedge dy \wedge dz$. Show that Δ is never involutive.

Problem 8. (15 pts)

Let $S \subset \mathbb{R}^3$ be the paraboloid

$$S = \{z = x^2 + y^2\}.$$

(a) Let g be the metric on S induced by the standard metric on $\mathbb{R}^3$. Show that g has the form

$$g = (1 + 4r^2)dr^2 + r^2 d\theta^2,$$

where (r, θ) are polar coordinates in (x, y).

(b) Find an orthonormal frame for this metric and the corresponding dual coframe.

(c) Show that the Gaussian curvature of S with this metric is $4/(1 + 4r^2)^2$.